Applications of Integration - Average Value, Centers of Mass

There are a number of applications of double and triple integration in 13.1 (Average Value) , 13.4 (Average Value) and 13.6 (Mass).
Read 13.6 (Lesson 23), review 13.1 and 13.4.

Suggested Homework:

13.6 - 7, 9, 11, 15, 17, 23, 29, 33, 35, 37

Average Value

DEFINITION Average Value of a Function over a Plane Region

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA.$$

DEFINITION Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of f over D is

$$\bar{f} = \frac{1}{\text{volume}(D)} \iiint_D f(x, y, z)\, dV.$$

Average Value Examples

Average coordinate values

Center of Mass

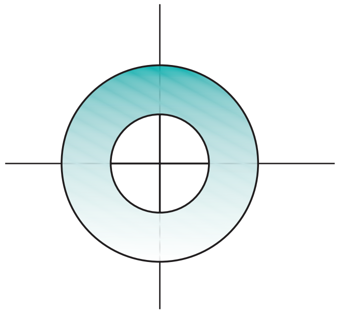
If a thin lamina of variable density $\delta(x, y)$ occupies a planar region R, the x and y coordinates of the **center of mass** are given by:

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_R x\delta(x, y)dA}{\iint_R \delta(x, y)dA}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\iint_R y\delta(x, y)dA}{\iint_R \delta(x, y)dA}$$

Example

- Find the center of mass of a thin washer occupying the region between circles of radius 1 and 2 centered at the origin, if the density is given by $\delta(x, y) = y + 2$.



Center of Mass in Three Dimensions

If a solid object with variable density $\delta(x, y, z)$ occupies a region R, the x and y and z coordinates of the **center of mass** are given by:

$$\overline{x} = \frac{M_{yz}}{M}, \qquad \overline{y} = \frac{M_{xz}}{M}, \qquad \overline{z} = \frac{M_{xy}}{M}$$

$$\overline{x} = \frac{\iiint_R x \cdot \delta(x, y, z)dV}{\iiint_R \delta(x, y, z)dV} \qquad \overline{y} = \frac{\iiint_R y \cdot \delta(x, y, z)dV}{\iiint_R \delta(x, y, z)dV}$$

$$\overline{z} = \frac{\iiint_R z \cdot \delta(x, y, z)dV}{\iiint_R \delta(x, y, z)dV}$$

Examples:

- Find the center of mass of the upper hemisphere of a ball of radius 3 centered at the origin. (Assume constant density.)
- Center of mass of the part of the ball $x^2 + y^2 + (z - 2)^2 \leq 4$ that lies inside the paraboloid $z = x^2 + y^2$